13 Theorems on Area

INTRODUCTION

We know that Geometry originated from the need of measuring land or recasting/refixing its boundaries in the process of distribution of certain land or field among different people. You may recall that the part of the plane enclosed by a simple closed curve is called a ***planar region*** corresponding to that figure and the magnitude (or measure) of this plane region is called the ***area*** of that figure. This magnitude of measure is always expressed with the help of a number (in some unit) such as 15 cm^2, 32 m^2, 3·5 hectares etc. So, we can say that the area of a simple closed plane figure is a number (in some unit) associated with the part of the plane enclosed by the figure.

In earlier classes, we have learnt some formulae for finding the areas of some different simple closed plane figures such as triangle, rectangle, square, parallelogram etc. In this chapter, we shall consolidate the knowledge about these formulae by studying some relationship between the areas of these geometric figures under the condition when they lie on the same base (or equal bases) and between the same parallel lines.

13.1 AXIOMS OF AREA

❑ Congruence area axiom

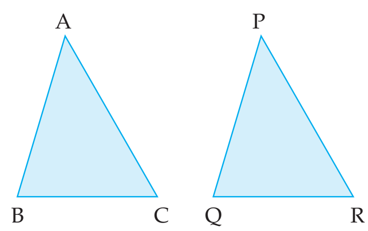
If two figures are congruent, then the areas enclosed by these figures are equal.

Thus, if two figures A and B are congruent, then area enclosed by A = area enclosed by B.

This is known as **congruence area axiom**.

In the adjoining figure, $\Delta ABC \cong \Delta PQR$.

So, area of ΔABC = area of ΔPQR.



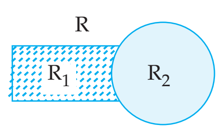
Note that by area of ΔABC we mean the area of the region enclosed by the triangle ABC.

❑ Addition area axiom

If a planar region R consists of two non-overlapping planar regions R_1 and R_2, then area of region R = area of region R_1 + area of region R_2.

This is known as **addition area axiom**.

In the adjoining figure, planar regions R_1 and R_2 are non-overlapping. If R is the total region *i.e.* the region made up of regions R_1 and R_2, then



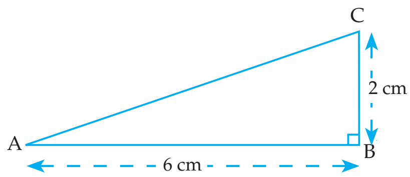
area of region R = area of region R_1 + area of region R_2.

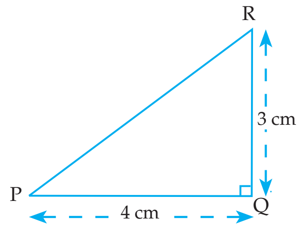
13.2 EQUAL FIGURES

Two figures are called equal if and only if they have equal area.

As two congruent figures have equal area, therefore, two congruence figures are always equal figures. However, the converse may not be true *i.e.* two equal figures may not be congruent.

For example, consider the two right triangles ABC and PQR given below:





Area of $\Delta ABC = \left(\frac{1}{2} \times 6 \times 2\right)$ cm^2 = 6 cm^2.

Area of $\Delta PQR = \left(\frac{1}{2} \times 4 \times 3\right)$ cm^2 = 6 cm^2.

$\therefore$ Area of ΔABC = area of ΔPQR

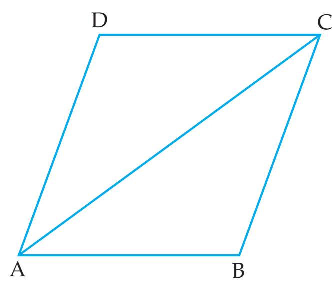
$\Rightarrow$ ΔABC and ΔPQR are equal figures.

Clearly, these triangles are not congruent.

13.3 THEOREMS ON AREA

Theorem 13.1

A diagonal of a parallelogram divides it into two triangles of equal areas.



Given. ABCD is a parallelogram and AC is its one diagonal.

To prove. Area of ΔABC = area of ΔACD.

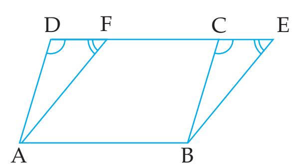
Proof.

Statements	*Reasons*
In ΔABC and ΔCDA	
1. AB = DC	1. Opp. sides of || gm ABCD.
2. BC = AD	2. Opp. sides of || gm ABCD.
3. AC = AC	3. Common.
4. $\Delta ABC \cong \Delta CDA$	4. SSS rule of congruency.
5. Area of ΔABC = area of ΔACD **Q.E.D.**	5. Congruence area axiom.

Theorem 13.2

Parallelograms on the same base and between the same parallel lines are equal in area.

Given. Two parallelograms ABCD and ABEF on the same base AB and between the same parallel lines AB and DE.

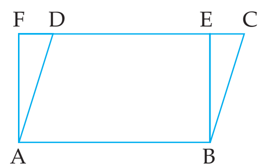


To prove. Area of || gm ABCD = area of || gm ABEF.

Proof.

Statements	*Reasons*
In ΔADF and ΔBCE	
1. $\angle$ADF = $\angle$BCE	1. Corres. $\angle$s, AD $\parallel$ BC and DE is a transversal.
2. $\angle$AFD = $\angle$BEC	2. Corres. $\angle$s, AF $\parallel$ BE and DE is a transversal.
3. AD = BC	3. Opp. sides of $\parallel$ gm ABCD.
4. ΔADF $\cong$ ΔBCE	4. AAS rule of congruency.
5. Area of ΔADF = area of ΔBCE	5. Congruent figures have equal area.
6. Area of ΔADF + area of quad. ABCF = area of ΔBCF + area of quad. ABCF	6. Adding same area on both sides.
7. Area of $\parallel$ gm ABCD = area of $\parallel$ gm ABEF	7. Addition area axiom.

Corollary 1. *A parallelogram and a rectangle on the same base and between the same parallel lines are equal in area.*



Proof. Let a parallelogram ABCD and a rectangle ABEF be on the same base AB and between the same parallel lines AB and FC (as shown in the adjoining figure).

We want to prove that area of $\parallel$ gm ABCD = area of rect. ABEF.

Since a rectangle is also a parallelogram, therefore,

area of $\parallel$ gm ABCD = area of rect. ABEF (Theorem 13.2).

Hence, a parallelogram and a rectangle on the same base and between the same parallel lines are equal in area.

Corollary 2. *Area of a parallelogram = base × height.*

Proof. By corollary 1, we have

area of parallelogram ABCD = area of rectangle ABEF ...(*i*)

(see figure of corollary 1)

Also area of rectangle ABEF = AB × BE ...(*ii*)

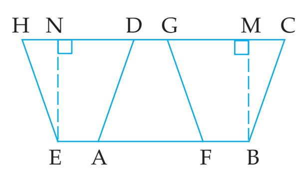
From (*i*) and (*ii*), we get

area of parallelogram ABCD = AB × BE = base × height.

Hence, the area of a parallelogram = base × height.

Corollary 3. *Parallelograms with equal bases and between the same parallels are equal in area.*

Proof. Let ABCD and EFGH be two parallelograms with equal bases *i.e.* AB = EF and between the same parallel lines EB and HC.



From B, draw BM $\perp$ DC and from E, draw EN $\perp$ HG, then BM = EN

($\because$ EBMN is a rectangle (why?)), so opp. sides are equal *i.e.* BM = EN)

By corollary 2,

area of parallelogram ABCD = base × height = AB × BM

= EF × EN ($\because$ AB = EF and BM = EN)

= area of parallelogram EFGH.

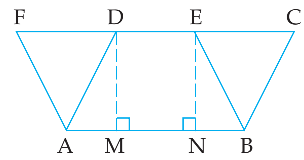
Hence, parallelograms with equal bases and between the same parallel lines are equal in area.

❑ Converse of theorem 13.2

The converse of the above theorem 13.2 is also true. In fact, we have:

Theorem 13.3

Parallelograms on the same base and having equal areas lie between same parallel lines.



Given. Two parallelograms ABCD and ABEF on the same base AB and area of || gm ABCD = area of || gm ABEF.

To prove. AB || FC.

Construction. From D, draw DM ⊥ AB and from E, draw EN ⊥ AB.

Proof.

Statements	*Reasons*
1. Area of || gm ABCD = AB × DM	1. Area of a || gm = base × height.
2. Area of || gm ABEF = AB × EN	2. Area of a || gm = base × height.
3. AB × DM = AB × EN ⇒ DM = EN	3. Area of || gm ABCD = area of || gm ABEF (given)
4. DM || EN	4. DM and EN are both perpendicular to the same line AB.
5. DMNE is a parallelogram.	5. Two sides DM and EN of quad. DMNE are equal and parallel.
6. MN || DE *i.e.* AB || FC	6. By definition of a || gm.

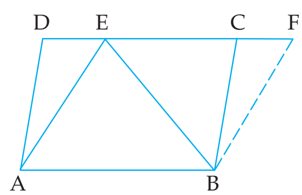
Corollary 1. *Parallelograms on equal bases and having equal areas have equal corresponding altitudes.*

(In the above proof of the theorem, we obtained DM = EN, so || gm ABCD and || gm ABEF have equal corresponding altitudes.)

Corollary 2. *Parallelograms on equal bases and having equal areas lie between same parallel lines.*

Theorem 13.4

Area of a triangle is half that of a parallelogram on the same base and between the same parallel lines.



Given. A triangle ABE and a parallelogram ABCD on the same base AB and between the same parallel lines AB and DC.

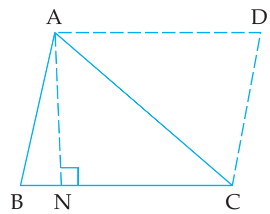
To prove. Area of ΔABE = $\frac{1}{2}$ area of || gm ABCD.

Construction. Through B, draw BF || AE to meet DC (produced) at F.

Proof.

Statements	*Reasons*
1. ABFE is a parallelogram	1. By construction.
2. Area of $\Delta ABE = \frac{1}{2}$ area of ‖ gm ABFE	2. BE is a diagonal of ‖ gm ABFE, and a diagonal divides it into two triangles of equal areas.
3. Area of ‖ gm ABCD = area of ‖ gm ABFE	3. Parallelograms on the same base and between the same parallels are equal in area.
4. Area of $\Delta ABE = \frac{1}{2}$ area of ‖ gm ABCD	4. From 2 and 3.

Corollary 1. *Area of a triangle* $= \frac{1}{2}$ *base × height*



Proof. Let ABC be a triangle with base BC and AN ⊥ BC, then height of ΔABC = AN. Through A and C draw lines parallel to BC and BA to meet at D, then ABCD is a parallelogram.

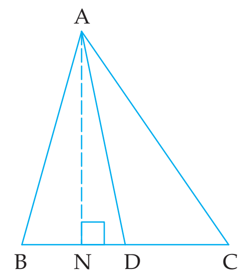
Thus, ΔABC and parallelogram ABCD are on the same base BC and between same parallel lines BC and AD.

Area of ΔABC $= \frac{1}{2}$ area of parallelogram ABCD (Theorem 13.4)

$= \frac{1}{2}$ BC × AN (area of ‖ gm = base × height)

$= \frac{1}{2}$ base × height.

Corollary 2. *A median of a triangle divides it into two triangles of equal areas.*



Proof. Let ABC be any triangle and AD be one of its medians (shown in the adjoining figure).

We need to show that area of ΔABD = area of ΔACD.

From A, draw AN ⊥ BC.

Now, area (ΔABD) $= \frac{1}{2}$ base × corresponding height

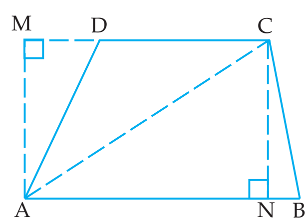
$= \frac{1}{2}$ BD × AN $= \frac{1}{2}$ DC × AN

(∵ D is mid-point of BC, so BD = DC)

= area of ΔACD.

Hence, a median of a triangle divides it into two triangles of equal area.

Corollary 3. *Area of a trapezium* $= \frac{1}{2}$ *(sum of parallel sides) × height.*



Proof. Let ABCD be a trapezium in which AB ‖ DC. Join AC. From C, draw CN ⊥ AB and from A, draw AM ⊥ CD (produced).

Then CN = AM = height of trapezium

= h (say).

Area of trapezium ABCD = area of ΔABC + area of ΔACD)

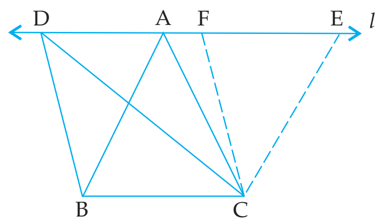
$= \frac{1}{2}$ AB × CN + $\frac{1}{2}$ DC × AM

$$= \frac{1}{2}(AB \times h + DC \times h) = \frac{1}{2}(AB + DC) \times h$$

$$= \frac{1}{2}(\text{sum of parallel sides}) \times \text{height.}$$

Theorem 13.5

Triangles on the same base (or equal bases) and between the same parallel lines are equal in area.



Given. Two triangles ABC and BCD on the same BC and between the same parallel lines BC and l.

To prove. Area of ΔABC = area of ΔBCD.

Construction. Through C, draw CE || BA and CF || BD to meet line l at E and F respectively.

Proof.

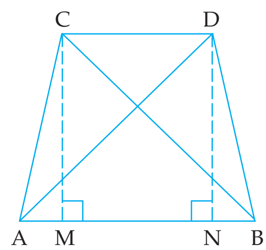
Statements	*Reasons*
1. ABCE is a parallelogram.	1. BC || AE (given), CE || BA (Const.)
2. BCFD is a parallelogram	2. BC || DF (given), CF || BD (Const.)
3. Area of || gm ABCE = area of || gm BCFD.	3. Parallelogram on same base and between same parallel lines.
4. Area of ΔABC = $\frac{1}{2}$ area of || gm ABCE	4. Area of a Δ is half that of a || gm on the same base and between same parallels.
5. Area of ΔBCD = $\frac{1}{2}$ area of || gm BCFD	5. Area of a Δ is half that of a || gm on the same base and between same parallels.
6. Area of ΔABC = area of ΔBCD	6. From 3, 4 and 5.

❑ Converse of theorem 13.5

The converse of the above theorem 13.5 is also true. In fact, we have:

Theorem 13.6

Triangles on the same base (or equal bases) and having equal areas lie between the same parallel lines.



Given. Two triangles ABC and ABD on the same base AB, and area of ΔABC = area of ΔABD.

To prove. CD || AB.

Construction. From C and D, draw perpendiculars CM and DN on AB respectively.

Proof.

Statements	*Reasons*
1. Area of ΔABC = $\frac{1}{2}$ AB × CM	1. Area of a triangle = $\frac{1}{2}$ base × height.
2. Area of ΔABD = $\frac{1}{2}$ AB × DN	2. Same as above.
3. $\frac{1}{2}$ AB × CM = $\frac{1}{2}$ AB × DN	3. Area of ΔABC = area of ΔABD (given)
4. CM = DN	4. From 3, cancelling $\frac{1}{2}$ AB.

5. CM ∥ DN	5. CM and DN are both perpendiculars to the same line AB.
6. CMND is a parallelogram.	6. Two sides CM and DN of quad. CMND are equal and parallel.
7. CD ∥ MN *i.e.* CD ∥ AB	7. By definition of a ∥ gm.

Corollary 1. *Triangles on the same base (or equal bases) and having equal areas have equal corresponding altitudes.*

(In the above proof of the theorem, we obtained CM = DN. So, ΔABC and ΔABD have equal corresponding altitudes.)

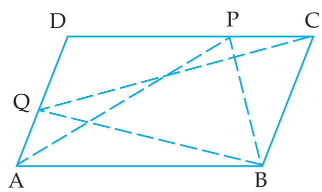
Corollary 2. *If two triangles lie between the same parallels (i.e. have equal altitudes), then the ratio of their areas equals the ratio of their bases.*

Corollary 3. *If two triangles have equal bases, then the ratio of their areas equals the ratio of their altitudes.*

Illustrative Examples

Example 1. *P and Q are any two points lying on the sides DC and AD respectively of a parallelogram ABCD. Show that area of ΔAPB = area of ΔBQC.*

Solution. Given a parallelogram ABCD, and P and Q are points lying on the sides DC and AD respectively as shown in the adjoining figure.



As ΔAPB and ∥ gm ABCD are on the same base and between the same parallels AB and DC,

$$\text{area of } \Delta APB = \frac{1}{2} \text{ area of } \parallel \text{gm } ABCD \qquad \ldots(i)$$

Also, as ΔBQC and ∥ gm ABCD are on the same BC and between the same parallels AD and BC,

$$\text{area of } \Delta BQC = \frac{1}{2} \text{ area of } \parallel \text{gm } ABCD \qquad \ldots(ii)$$

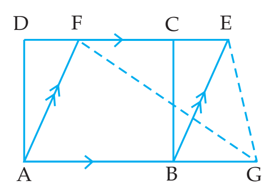
From (*i*) and (*ii*), we get

area of ΔAPB = area of ΔBQC.

Example 2. *In the adjoining figure, ABCD is a rectangle with sides AB = 8 cm and AD = 5 cm. Compute*

(i) area of parallelogram ABEF

(ii) area of ΔEFG.



Solution. (*i*) Area of ∥ gm ABEF

= area of rectangle ABCD

(on the same base AB and between the same parallels AB and DE)

= (8×5) cm^2 = 40 cm^2.

(*ii*) Area of ΔEFG = $\frac{1}{2}$ area of ∥ gm ABEF

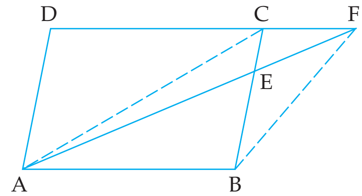
(on the same base FE and between the same parallels FE and AG)

$= \left(\frac{1}{2} \times 40\right)$ cm^2 = 20 cm^2.

Example 3. *A point E is taken on the side BC of a parallelogram ABCD. AE and DC are produced to meet at F. Prove that*

area of ΔADF = area of quad. ABFC.

Solution. Join AC and BF.



As triangles ACF and BCF have same base CF and are between the same parallels AB and CF ($\because$ AB || DC),

area of ΔACF = area of ΔBCF ...(*i*)

As diagonal AC divides || gm ABCD into two triangles of equal area,

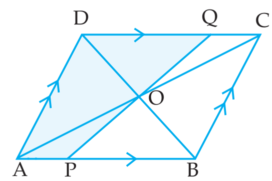
area of ΔDAC = area of ΔABC ...(*ii*)

On adding (*i*) and (*ii*), we get

area of ΔACF + area of ΔDAC = area of ΔBCF + area of ΔABC

$\Rightarrow$ area of ΔADF = area of quad. ABFC.

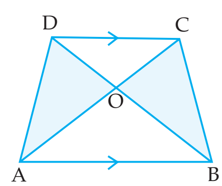
Example 4. *The diagonals of a parallelogram ABCD intersect at O. A straight line through O meets AB at P and the opposite side CD at Q. Prove that area of quad. APQD = $\frac{1}{2}$ area of || gm ABCD.*



Proof.

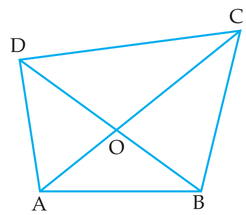
Statements	*Reasons*
1. Area of $\Delta ACD = \frac{1}{2}$ area of || gm ABCD	1. Diagonal divides a ||gm into two Δs of equal area.
In ΔOAP and ΔOCQ	
2. $\angle OAP = \angle OCQ$	2. Alt. $\angle$s.
3. $\angle AOP = \angle COQ$	3. Vert. opp. $\angle$s.
4. AO = OC	4. Diagonals bisect each other.
5. $\Delta OAP \cong \Delta OCQ$	5. ASA rule of congruency.
6. Area of ΔOAP = area of ΔOCQ	6. Congruence area axiom.
7. Area of ΔOAP + area of quad. AOQD = area of ΔOCQ + area of quad. AOQD	7. Adding same area on both sides.
8. Area of quad. APQD = area of ΔACD	8. Addition area axiom.
9. Area of quad. APQD $= \frac{1}{2}$ area of || gm ABCD **Q.E.D.**	9. From 8 and 1.

Example 5. *ABCD is a trapezium with AB || DC, and diagonals AC and BD meet at O. Prove that area of ΔDAO = area of ΔOBC.*



Proof.

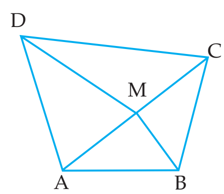
Statements	*Reasons*
1. AB || DC	1. Given.
2. Area of ΔABD = area of ΔABC	2. Δs on the same base AB and between the same parallels AB and CD are equal in area.
3. Area of ΔDAO + area of ΔOAB = area of ΔOBC + area of ΔOAB	3. Addition area axiom.
4. Area of ΔDAO = area of ΔOBC **Q.E.D.**	4. Subtracting same area from both sides.



Example 6. *The diagonals AC and BD of a quadrilateral ABCD intersect at O. If OB = OD, prove that the triangles ABC and ACD are equal in area.*

Proof.

Statements	*Reasons*
1. AO is median of ΔABD	1. OB = OD (given).
2. Area of ΔOAB = area of ΔOAD	2. Median divides a Δinto two Δs of equal area.
3. CO is median of ΔCBD	3. OB = OD (given)
4. Area of ΔOBC = area of ΔOCD	4. Median divides a Δ into two Δs of equal area.
5. Area of ΔOAB + area of ΔOBC = area of ΔOAD + area of ΔOCD	5. Adding 2 and 4
6. Area of ΔABC = area of ΔACD **Q.E.D.**	6. Addition area axiom.

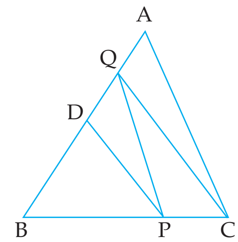


Example 7. *In quadrilateral ABCD, M is mid-point of the diagonal AC. Prove that area of quad. ABMD = area of quad. DMBC.*

Proof.

Statements	*Reasons*
1. BM is median of ΔBCA	1. M is mid-point of AC (given).
2. Area of ΔABM = area of ΔMBC	2. Median divides a Δinto two Δs of equal area.
3. DM is median of ΔDAC	3. M is mid-point of AC (given).
4. Area of ΔDAM = area of ΔDMC	4. Median divides a triangle into two Δs of equal area.
5. Area of ΔABM + area of ΔDAM = area of ΔMBC + area of ΔDMC	5. Adding 2 and 4.
6. Area of quad. ABMD = area of quad. DMBC **Q.E.D.**	6. Addition area axiom.

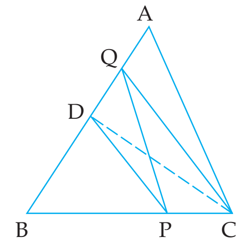
Example 8. *In the adjoining figure, D is mid-point of AB and P is any point on side BC of ΔABC. If CQ || PD meets AB in Q, then prove that area of ΔBPQ =* $\frac{1}{2}$ *area of ΔABC.*



Solution. Join CD.

Triangles DPQ and DPC are on the same base PD and between same parallels CQ and PD,

∴ area of ΔPDQ = area of ΔDPC ...(*i*)



As D is mid-point of AB, so CD is a median of ΔABC. Since a median divides a triangle into two triangles of equal area,

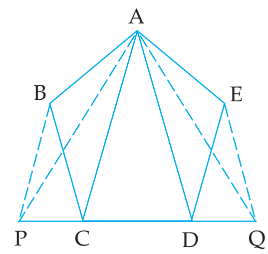
area of ΔBCD = $\frac{1}{2}$ area of ΔABC

⇒ area of ΔBPD + area of ΔDPC = $\frac{1}{2}$ area of ΔABC

⇒ area of ΔBPD + area of ΔDPQ = $\frac{1}{2}$ area of ΔABC (using (*i*))

⇒ area of ΔBPQ = $\frac{1}{2}$ area of ΔABC.

Example 9. *In the adjoining figure, ABCDE is any pentagon. BP drawn parallel to AC meets DC produced at P and EQ drawn parallel to AD meets CD produced at Q. Prove that area of ABCDE = area of ΔAPQ.*



Solution. ΔPCA and ΔBCA are on the same base CA and between same parallels BP || AC.

∴ area of ΔBCA = area of ΔPCA ...(*i*)

ΔEAD and ΔQAD are on the same base AD and between same parallels EQ || AD,

∴ area of ΔEAD = area of ΔQAD ...(*ii*)

Also, area of ΔACD = area of ΔACD ...(*iii*)

On adding (*i*), (*iii*) and (*ii*), we get

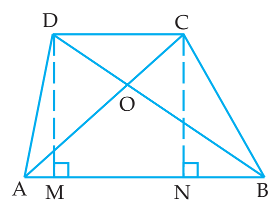
area of ΔBCA + area of ΔACD + area of ΔEAD

= area of ΔPCA + area of ΔACD + area of ΔQAD

⇒ area of ABCDE = area of ΔAPQ.

Example 10. *The diagonals AC and BD of a quadrilateral ABCD intersect at O in such a way that area of ΔOAD = area of ΔOBC. Prove that ABCD is a trapezium.*

Solution. Draw DM ⊥ AB and CN ⊥ AB.



As DM and CN are both perpendiculars to AB, therefore, DM || CN.

Given area of ΔOAD = area of ΔOBC

⇒ area of ΔOAD + area of ΔOAB

= area of ΔOBC + area of ΔOAB (adding same area on both sides)

⇒ area of ΔABD = area of ΔABC

⇒ $\frac{1}{2}AB \times DM = \frac{1}{2}AB \times CN$

⇒ DM = CN.

Thus DM || CN and DM = CN, therefore, DMNC is a parallelogram

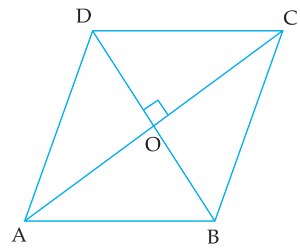
⇒ DC || MN *i.e.* DC || AB.

Hence, ABCD is a trapezium.

Example 11. *Prove that area of a rhombus =* $\frac{1}{2}$ $\times$ *product of diagonals.*

Solution. Let ABCD be a rhombus, and let its diagonals intersect at O.

Since the diagonals of a rhombus cut at right angles, OB $\perp$ AC and OD $\perp$ AC.



As area of a triangle = $\frac{1}{2}$ base $\times$ height,

$\therefore$ area of ΔABC = $\frac{1}{2}$ AC $\times$ OB ...(*i*)

and area of ΔACD = $\frac{1}{2}$ AC $\times$ OD ...(*ii*)

On adding (*i*) and (*ii*), we get

area of ΔABC + area of ΔACD = $\frac{1}{2}$ AC $\times$ OB + $\frac{1}{2}$ AC $\times$ OD

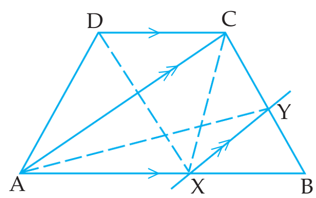
$\Rightarrow$ area of rhombus ABCD = $\frac{1}{2}$ AC $\times$ (OB + OD)

= $\frac{1}{2}$ AC $\times$ BD

= $\frac{1}{2}$ $\times$ product of diagonals.

Example 12. *ABCD is a trapezium with AB || DC. A line parallel to AC intersects AB at X and BC at Y. Prove that: area of ΔADX = area of ΔACY.*

Solution. Join CX.



As triangles ADX and ACX have same base AX and are between the same parallels (AB || DC given, so, AX || DC),

$\therefore$ area of ΔADX = area of ΔACX ...(*i*)

As triangles ACY and ACX have same base AC and are between the same parallels (XY || AC given),

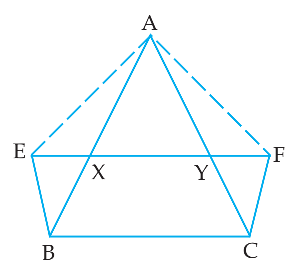
$\therefore$ area of ΔACY = area of ΔACX ...(*ii*)

From (*i*) and (*ii*), we get

area of ΔADX = area of ΔACY.

Example 13. *XY is a line parallel to side BC of a triangle ABC. If BE || CA and FC || AB meet XY at E and F respectively, show that area of ΔABE = area of ΔACF.*

Solution. As ΔABE and || gm EBCY have the same base BE and are between the same parallels BE || CA (given),



$\therefore$ area of ΔABE = $\frac{1}{2}$ area of || gm EBCY ...(*i*)

As ΔACF and || gm XBCF have the same base CF and are between the same parallels FC || AB (given),

$\therefore$ area of ΔACF = $\frac{1}{2}$ area of || gm XBCF ...(*ii*)

But || gm EBCY and || gm XBCF have the same base BC and are between the same parallels (XY || BC given),

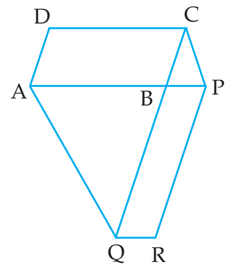
$\therefore$ area of || gm EBCY = area of || gm XBCF

$\Rightarrow$ $\frac{1}{2}$ area of || gm EBCY = $\frac{1}{2}$ area of || gm XBCF

$\Rightarrow$ area of ΔABE = area of ΔACF (using (*i*) and (*ii*))

Example 14. *In the adjoining figure, the side AB of a parallelogram ABCD is produced to any point P. A line through A and parallel to CP meets CB produced at Q and then parallelogram PBQR is completed. Show that*

area of || gm ABCD = area of || gm PBQR.



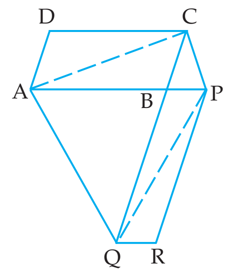
Solution. Join AC and PQ.

As AC is a diagonal of || gm ABCD,

area of || gm ABCD = 2 area of ΔABC ...(*i*)

As PQ is a diagonal of || gm PBQR,

area of || gm PBQR = 2 area of ΔPBQ ...(*ii*)



Now, triangles CAQ and PAQ have the same base AQ and are between the same parallels AQ || CP,

$\therefore$ area of ΔCAQ = area of ΔPAQ

$\Rightarrow$ area of ΔCAQ – area of ΔBAQ = area of ΔPAQ – area of ΔBAQ

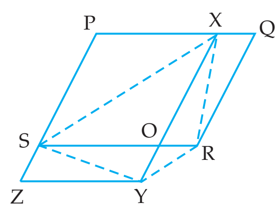
(subtracting same area from both sides)

$\Rightarrow$ area of ΔABC = area of ΔPBQ

$\Rightarrow$ 2 area of ΔABC = 2 area of ΔPBQ

$\Rightarrow$ area of || gm ABCD = area of || gm PBQR (using (*i*) and (*ii*))

Example 15. *In the adjoining figure, PQRS and PXYZ are two parallelograms of equal area. Prove that SX is parallel to YR.*



Solution. Join XR, SY.

Given area of || gm PQSR = area of || gm PXYZ.

Subtract area of || gm PSOX from both sides.

$\therefore$ Area of || gm XORQ = area of || gm SZYO

$\Rightarrow$ area of Δ XOR = area of ΔSYO

(because diagonal divides a || gm into two equal areas)

Adding area of ΔOYR to both sides, we get

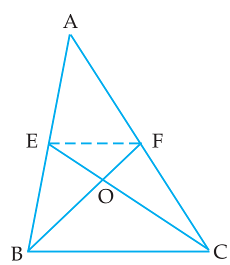
area of ΔXYR = area of ΔSYR.

Also the Δs XYR and SYR have the same base YR, therefore, these lie between the same parallels

$\Rightarrow$ SX is parallel to YR.

Example 16. *E and F are mid-points of the sides AB and AC respectively of a triangle ABC. If BF and CE meet at O, prove that area of ΔOBC = area of quad. AEOF.*

Solution. Join EF.



As E and F are mid-points of AB and AC respectively, EF || BC.

$\therefore$ Area of ΔEBC = area of ΔFBC.

(Triangles on the same base BC and between same parallels)

$\Rightarrow$ area of ΔEBC – area of ΔOBC

= area of ΔFBC – area of ΔOBC

$\Rightarrow$ area of ΔBOE = area of ΔCOF ...(*i*)

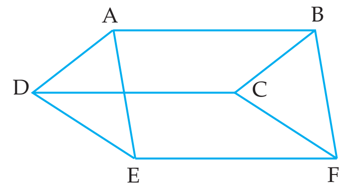
As F is mid-point of AC, area of ΔFBC = area of ΔABF

($\because$ A median divides a triangle into two triangles of equal area).

$\Rightarrow$ area of ΔFBC – area of ΔCOF = area of ΔABF – area of ΔBOE [using (*i*)]

$\Rightarrow$ area of ΔOBC = area of quad. AEOF (from figure)

Example 17. *In the adjoining figure, ABCD, DCFE and ABFE are parallelograms. Show that area of ΔADE = area of ΔBCF.*



Solution. As ABCD is a parallelogram,

AD = BC (opp. sides of a || gm)

Similarly, DE = CF and AE = BF.

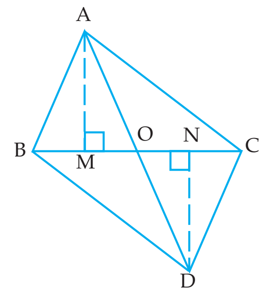
In ΔADE and ΔBCF,

AD = BC, DE = CF and AE = BF

∴ ΔADE ≅ ΔBCF (by SSS rule of congruency)

∴ area of ΔADE = area of ΔBCF (congruent figures have equal areas)

Example 18. *Triangles ABC and DBC are on the same base BC with A, D on opposite sides of BC. If area of ΔABC = area of ΔDBC, prove that BC bisects AD.*



Solution. Let BC and AD intersect at O.

Draw AM ⊥ BC and DN ⊥ BC.

Given area of ΔABC = area of ΔDBC

$\Rightarrow \quad \frac{1}{2} BC \times AM = \frac{1}{2} BC \times DN$

$\Rightarrow \quad AM = DN.$

In ΔAMO and ΔDNO,

∠AOM = ∠DON (vert. opp. ∠s)

∠AMO = ∠DNO (each angle = 90°)

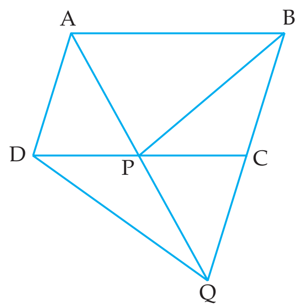
AM = DN (proved above)

∴ ΔAMO ≅ ΔDNO (AAS rule of congruency)

∴ AO = DO (c.p.c.t.)

Hence, BC bisects AD.

Example 19. *In the adjoining figure, ABCD is a parallelogram and BC is produced to a point Q such that CQ = AD. If AQ intersects DC at P, show that area of ΔBPC = area of ΔDPQ.*



Solution. Join AC.

As triangles BPC and APC have same base PC and are between the same parallels (AB || DC *i.e.* AB || PC),

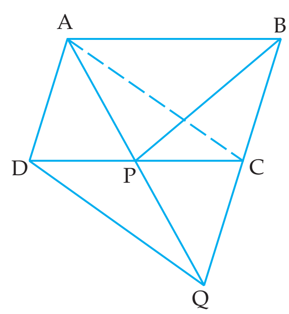
∴ area of ΔBPC = area of ΔAPC ...(*i*)

In quad. ADQC, AD || CQ

(∵ AD || BC, opp. sides of || gm ABCD)

AD = CQ (given)

∴ ADQC is a parallelogram, so its diagonals AQ and DC bisect each other *i.e.* DP = PC and AP = PQ.



In ΔAPC and ΔQPD,

PC = DP

AP = PQ

∠APC = ∠QPD (vert. opp. ∠s)

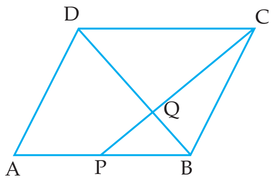
ΔAPC ≅ ΔQPD

∴ area of ΔAPC = area of ΔDPQ ...(*ii*)

From (*i*) and (*ii*), we get

area of ΔBPC = area of ΔDPQ.

Example 20. *In the adjoining figure, ABCD is a parallelogram. P is mid-point of AB and CP meets the diagonal BD at Q. If area of Δ PBQ = 10 cm^2, calculate*



(i) PQ : QC

(ii) area of ΔPBC

(iii) area of parallelogram ABCD.

Solution. (*i*) Since P is mid-point of AB, $PB = \frac{1}{2} AB$.

But AB = DC ($\because$ ABCD is a || gm) $\Rightarrow PB = \frac{1}{2} DC$ ...(1)

In ΔPBQ and ΔCDQ,

$\angle PQB = \angle DQC$ (vert. opp. $\angle$s) and $\angle PBQ = \angle QDC$ (alt. $\angle$s)

$\Rightarrow \Delta PBQ \sim \Delta CDQ$

$\therefore \frac{PQ}{QC} = \frac{PB}{DC} = \frac{1}{2}$ (using (1))

$\Rightarrow$ PQ : QC = 1 : 2.

(*ii*) PQ : QC = 1 : 2 $\Rightarrow$ PQ : PC = 1 : 3 ...(2)

Since the bases CP, QP of Δs PBC, PBQ lie along the same line, and these triangles have equal heights, therefore,

$$\frac{\text{area of } \Delta PBC}{\text{area of } \Delta PBQ} = \frac{PC}{PQ} = \frac{3}{1} \quad \text{(using (2))}$$

$\Rightarrow$ area of ΔPBC = 3 $\times$ area of ΔPBQ = (3 $\times$ 10) cm^2 = 30 cm^2.

(*iii*) Area of ΔABC = 2 $\times$ area of ΔPBC

($\because$ median of a triangle divides it into two triangles of equal areas)

= (2 $\times$ 30) cm^2 = 60 cm^2.

Area of || gm ABCD = 2 $\times$ area of ΔABC.

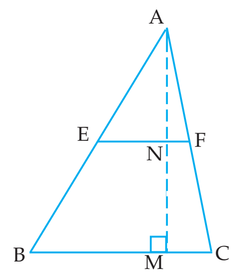
($\because$ diagonal divides a || gm into two triangles of equal areas)

= (2 $\times$ 60) cm^2 = 120 cm^2.

Example 21. *ABC is a triangle whose area is 50 cm^2. E and F are mid-points of the sides AB and AC respectively. Prove that EBCF is a trapezium. Also find its area.*

Solution. Since E and F are mid-points of the sides AB and AC respectively,

EF || BC and $EF = \frac{1}{2} BC$.



As EF || BC, EBCF is a trapezium.

From A, draw AM $\perp$ BC.

Let AM meet EF at N.

Since EF || BC, $\angle ENA = \angle BMN$.

But $\angle BMN = 90°$ ($\because$ AM $\perp$ BC)

so $\angle ENA = 90°$ *i.e.* AN $\perp$ EF.

Also, as E is mid-point of AB and EN || BM, N is mid-point of AM.

Now, area of $\Delta AEF = \frac{1}{2} EF \times AN = \frac{1}{2}(\frac{1}{2} BC \times \frac{1}{2} AM)$

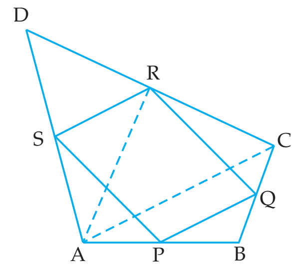
$= \frac{1}{4}(\frac{1}{2} BC \times AM) = \frac{1}{4}$(area of ΔABC)

$= \frac{1}{4}(50 \text{ cm}^2) = 12{\cdot}5 \text{ cm}^2$.

$\therefore$ Area of trapezium EBCF = area of ΔABC – area of ΔAEF

= 50 cm^2 – 12·5 cm^2 = 37·5 cm^2.

Example 22. *Prove that the area of the quadrilateral formed by joining the mid-points of the adjacent sides of a quadrilateral is half the area of the given quadrilateral.*



Given. A quadrilateral ABCD, and PQRS is the quadrilateral formed by joining mid-points of the sides AB, BC, CD and DA respectively.

To prove. Area of quad. PQRS = $\frac{1}{2}$ area of quad. ABCD.

Construction. Join AC and AR.

Proof.

Statements	*Reasons*
1. Area of ΔARD = $\frac{1}{2}$ area of ΔACD	1. Median divides a triangle into two triangles of equal area.
2. Area of ΔSRD = $\frac{1}{2}$ area of ΔARD	2. Same as in 1.
3. Area of ΔSRD = $\frac{1}{4}$ area of ΔACD	3. From 1 and 2.
4. Area of ΔPBQ = $\frac{1}{4}$ area of ΔABC	4. As in 3.
5. Area of ΔSRD + area of ΔPBQ = $\frac{1}{4}$(area of ΔACD + area of ΔABC)	5. Adding 3 and 4.
6. Area of ΔSRD + area of ΔPBQ = $\frac{1}{4}$ area of quad. ABCD	6. Addition area axiom.
7. Area of ΔAPS + area of ΔQCR = $\frac{1}{4}$ area of quad. ABCD	7. Same as in 6.
8. Area of ΔAPS + area of ΔPBQ + area of ΔQCR + area of ΔSRD = $\frac{1}{2}$ area of quad. ABCD	8. Adding 6 and 7.
9. Area of ΔAPS + area of ΔPBQ + area of ΔQCR + area of ΔSRD + area of quad. PQRS = area of quad. ABCD	9. Addition area axiom.
10. Area of quad. PQRS = $\frac{1}{2}$ area of quad. ABCD **Q.E.D.**	10. Subtracting 8 from 9.

Exercise 13

1. Prove that the line segment joining the mid-points of a pair of opposite sides of a parallelogram divides it into two equal parallelograms.

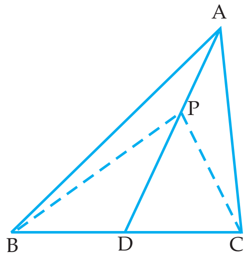
2. Prove that the diagonals of a parallelogram divide it into four triangles of equal area.

3. (*a*) In the figure (1) given below, AD is median of ΔABC and P is any point on AD. Prove that

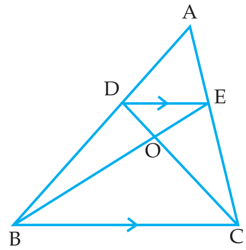
(*i*) area of ΔPBD = area of ΔPDC (*ii*) area of ΔABP = area of ΔACP.

(*b*) In the figure (2) given below, DE || BC. Prove that

(*i*) area of ΔACD = area of ΔABE (*ii*) area of ΔOBD = area of ΔOCE.



(1)



(2)

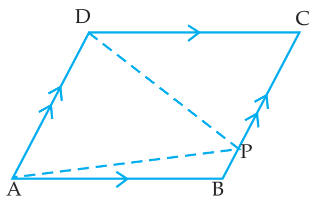
Hint: (*b*) (*i*) Area of ΔDEC = area of ΔDEB, add area of ΔADE to both sides.

4. (*a*) In the figure (1) given below, ABCD is a parallelogram and P is any point in BC. Prove that, area of ΔABP + area of ΔDPC = area of ΔAPD.

(*b*) In the figure (2) given below, O is any point inside a parallelogram ABCD. Prove that

(*i*) area of ΔOAB + area of ΔOCD = $\frac{1}{2}$ area of || gm ABCD.

(*ii*) area of ΔOBC + area of ΔOAD = $\frac{1}{2}$ area of || gm ABCD.



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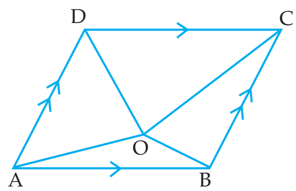


(2)

Hint: (*b*) (*i*) Through O, draw a straight line parallel to AB.

5. If E, F, G and H are mid-points of the sides AB, BC, CD and DA respectively of a parallelogram ABCD, prove that area of quad. EFGH = $\frac{1}{2}$ area of ||gm ABCD.

Hint: Join HF. AH = $\frac{1}{2}$ AD and BF = $\frac{1}{2}$ BC ⇒ AH = BF and AH || BF, so ABFH is a ||gm.

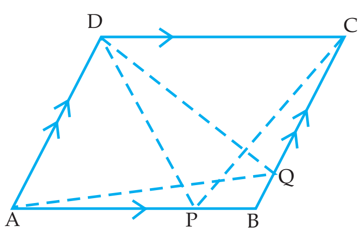
∴ Area of ΔEFH = $\frac{1}{2}$ area of || gm ABFH.

6. (*a*) In the figure (1) given below, ABCD is a parallelogram. P, Q are any two points on the sides AB and BC respectively. Prove that

area of ΔCPD = area of ΔAQD.

(*b*) In the figure (2) given below, PQRS and ABRS are parallelograms and X is any point on the side BR. Show that

area of ΔAXS = $\frac{1}{2}$ area of || gm PQRS.



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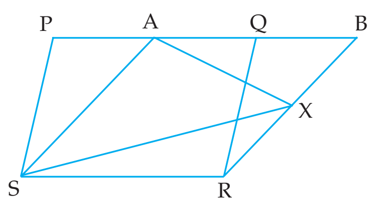

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